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ON STABILITY OF AN AUTONOMOUS HAMILTONIAN SYSTEM WITH TWO DEGREES OF FREEDOM UNDER FIRST-ORDER RESONANCE

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A stability problem for the equilibrium positions of autonomous Hamiltonian systems of ordinary differential equations with two degrees of freedom is examined for the case when one of the frequencies of the linear system equals zero. The stability question is resolved in a nonlinear formulation. The cases of simple (Kamenkov's case) and of multiple (Liapunov's case) elementary divisors of the matrix defining the linear system are studied. The stability or instability of the equilibrium position, depending on the coefficients of the Hamiltonian function, is proved.

1. We consider an autonomous Hamiltonian system with two degrees of freedom. The coordinates x_1 and x_2 and the moments X_1 and X_2 are chosen in such a way that the origin of the phase space coincides with the equilibrium position of the differential equation system and the Hamiltonian function is written as a series

$$H = H_2 + \ldots + H_m + \ldots \tag{1.1}$$

$$H_m = \sum_{\nu_1+\mu_1+\nu_2+\mu_2=m} h_{\nu_1\mu_1\nu_2\mu_2} x_1^{\nu_1} X_1^{\mu_1} x_2^{\nu_2} X_2^{\mu_2} \tag{1.2}$$

where H_m are uniform polynomials of degree m in the coordinates and momenta If H_2 is a sign-definite function of its variables, then the equilibrium position is stable by Liapunov's theorem [1]. If H_2 is not a sign-definite function, but stability in the first approximations obtains and the frequencies ω_1 and ω_2 $(0 \leqslant \omega_1 \leqslant \omega_2)$ of the linear system are not related by resonance relations up to fourth order, inclusively, then in the majority of cases the stability question is resolved by the Arnold-Moser theorem [2, 3]. Suppose that integers n_1 and n_2 exist such that $0 < |n_1| + |n_2| \leqslant 4$ and $n_1\omega_1 + n_2\omega_2 = 0$. Then the Arnold-Moser theorem is inapplicable and the stability problem requires a special investigation. Stability under resonance of third $(2\omega_1 = \omega_2)$ and fourth $(3\omega_1 = \omega_2)$ orders have been investigated in [4, 5]. Stability under a second-order $(\omega_1 = \omega_2)$ resonance was analyzed in [6].

The aim of the present paper is to obtain conditions for stability and instability under first-order resonance, i.e. when one of the frequencies of the linear system equals zero: $\omega_1 = 0$ and $\omega_2 \neq 0$. At first we turn to the question of the normal form of the quadratic part of the Hamiltonian function (1.1). In the most general form such a normal form has been established by Williamson and by D. M. Galin and is given in [7].

Let a linear system with Hamiltonian H_2 (x_1, X_1, x_2, X_2) be given. First of all we note that since $\omega_1 \neq \omega_2$ and $\omega_2 \neq 0$, the Hamiltonian can be reduced to a real canonic transformation to [8]

$$H_2 = h_2 + {}^1/_2\delta_2\omega_2 (x_2^2 + X_2^2) \quad (\delta_2 = \pm 1)$$

where h_2 depends only on the variables x_1 and X_1 and in the most general case is

$$h_2 = {}^1/_2 a x_1^2 + b x_1 X_1 + {}^1/_2\, c X_1^2$$

The equations of motion of such a system with one degree of freedom are

$$\dot{\xi} = Jh\xi$$

$$\xi = \begin{Vmatrix} x_1 \\ X_1 \end{Vmatrix}, \quad J = \begin{Vmatrix} 0 & 1 \\ -1 & 0 \end{Vmatrix}, \quad h = \begin{Vmatrix} a & b \\ b & c \end{Vmatrix}$$

The defining equation of this system $\det (Jh - \sigma E) = 0$, where E is the unit matrix, then takes the form

$$\sigma^2 + D = 0 \quad (D = \det (Jh) = ac - b^2)$$

It is obvious that if the defining equation has a zero root, $D = 0$. It can be shown that all cases can be reduced to two when $D = 0$ by linear real canonic substitutions: (1) $a = b = c = 0$; (2) $a \neq 0$ and $b = c = 0$. In the first and the second cases the invariant polynomials of the defining matrix $Jh - \sigma E$ equal, respectively: (1) $i_1 = i_2 = \sigma$; (2) $i_1 = \sigma^2$ and $i_2 = 1$. Consequently, in the first case the defining matrix has simple elementary divisors, while in the second case, multiple ones. We note in addition that in the first case $\operatorname{rank} (Jh) = 0$, and in the second, $\operatorname{rank} (Jh) = 1$. Thus in the first case $h_2 \equiv 0$ and the final normal form of the function H_2 is

$$H_2 = {}^1/_2 \delta_2 \omega_2 (x_2^2 + X_2^2) \qquad (1.3)$$

In the case of multiple elementary divisors we can always achieve $a = \delta_1 = \pm 1$ using canonic substitutions, i.e. we can assume that

$$H_2 = {}^1/_2 \delta_1 x_1^2 + {}^1/_2 \delta_2 \omega_2 (x_2^2 + X_2^2) \qquad (1.4)$$

The stability problem for a non-Hamiltonian system of two differential equations in the case of a pair of zero roots of the defining equation and multiple elementary divisors (the Liapunov case) was exhaustively considered in [9]. The author of [10] considered an analogous problem, but for the case of simple divisors (the Kamenkov case). In addition, Kamenkov considered the stability problem for systems of many differential equations with a pair of zero roots. However, Kamenkov assumed that all the remaining roots of the defining equation had negative real parts. It is clear that such a situation can never arise in a Hamiltonian system, and, since Kamenkov's proofs are based essentially on the assumption mentioned, there is a clear need for a special analysis of the case of Hamiltonian systems (or the case with pure imaginary roots). Nevertheless, certain ideas from the proof of instability, worked out by Liapunov and Kamenkov for systems with one degree of freedom, can be used in the investigation of systems with a larger number of degrees of freedom.

2. In almost all the nonresonance and resonance stability problems, analyzed to date, for canonic systems with two degrees of freedom the investigation is carried out along the following lines (an exception is the case of multiple elementary divisors with $\omega_1 = \omega_2$ [6]). The Hamiltonian function is reduced to normal form (each to its own). Next, using the integral $H = H^\circ = \text{const}$, where H° is a small number or zero, a reduction is effected, i.e. the investigation is reduced to a system with one degree of freedom, but a

nonautonomous one. If the Hamiltonian function of the resulting system turns out to be sign-variable, the instability of the original system is proved by any means; if sign-definite, the stability is proved. In the case when the first term in the expansion of the Hamiltonian function of the system with one degree of freedom is of constant sign, the stability question is resolved by terms of next order. Sign-variability is possible only in the resonance situation. Taking all these preliminary discussions into account, we can formulate the following fundamental theorem the use of which enables us to resolve the stability question for two-frequency systems in almost all cases.

Theorem 2.1. Let the Hamiltonian function of the system with two degrees of freedom, as a result of reduction, be

$$K = r^{\alpha}\Phi(\varphi) + K^{*}(r, \varphi, t, H^{\circ}) \tag{2.1}$$

$$\alpha > 1, \quad K^{*} = O(r^{\alpha+\alpha_1}), \quad (\alpha_1 > 0)$$

where the function Φ and K^{*} are τ-periodic in φ, while, in addition, K^{*} is 2π-periodic in t. In (2.1) H° is a sufficiently small value of the constant energy. Then, if the equation $\Phi(\varphi) = 0$ does not have real roots when $0 \leqslant \varphi \leqslant \tau$, the equilibrium position is Liapunov-stable. If a number φ^{*} exists such that $\Phi(\varphi^{*}) = 0$ but $\Phi'(\varphi^{*}) \neq 0$, the equilibrium position is unstable.

Note 2.1. If for all roots φ^{**} of the equation $\Phi(\varphi) = 0$ it happens that $\Phi'(\varphi^{**}) = 0$, the stability question is resolved by high-order terms.

Note 2.2. Theorem 2.1 is a simple generalization of Theorem 2.1 from [6], into which it turns when $\alpha = 2$ and $\tau = 2\pi$.

At first we prove an assertion on instability using Chetaev's theorem [11]. We note that from the periodicity condition on the function $\Phi(\varphi)$ and from the fact that $\Phi'(\varphi) \neq 0$ follows the possibility of selecting a root φ^{*} of the equation $\Phi(\varphi) = 0$ such that $\Phi'(\varphi^{*}) < 0$. As the Chetaev function we take

$$V = r \sin \Psi, \quad \Psi = \frac{\pi}{2a}(\varphi - \varphi^{*} + a)$$

where the fairly small (but finite) number a is chosen such that there are no other zeros of the function Φ in the neighborhood $\varphi^{*} - a \leqslant \varphi \leqslant \varphi^{*} + a$, while Φ' retains its sign in this neighborhood. The total derivative of the Chetaev function, taken relative to the equations of motion with the Hamiltonian function (2.1), is

$$\frac{dV}{dt} = r^{\alpha}\left\{\alpha\frac{\pi}{2a}\Phi\cos\Psi - \Phi'\sin\Psi\right\} + O(r^{\alpha+\alpha_1}) \tag{2.2}$$

It is easy to see that the function (2.2) is always positive in the region $V \geqslant 0$ and, consequently, the equilibrium position is unstable.

To prove stability we pass to the variables "action I — angle W" by the formulas $r = \partial S / \partial\varphi$ and $W = \partial S / \partial I$, where the generating function of the canonic transformation $r, \varphi \to I, W$ equals

$$S(I, \varphi) = I\tau\frac{E(\varphi)}{E(\tau)}, \quad E(u) = \int_0^u [\Phi(\varphi)]^{1/\alpha}\, d\varphi$$

Note that the integral $E(u)$ always exists for $0 \leqslant u \leqslant \tau$ and when the theorem's hypotheses are satisfied. Then, in the new variables the Hamiltonian (2.1) is written as

$$K(I, W) = \gamma(I) + K^{*}(I, W, t, H^{\circ}), \quad \gamma(I) = [I\tau / E(\tau)]^{\alpha}$$

where the fairly smooth function K^* has the period τ in φ, the period 2π in t and an order not lower than $\alpha + \alpha_1$ $(\alpha_1 > 0)$ relative to I. Since $d\gamma / dI \neq 0$ in the ring $0 < c_1 \leqslant I \leqslant c_2$, the equilibium position being investigated is stable by virtue of Moser's theorem on invariant curves [3].

3. Suppose that the defining equation of the system with Hamiltonian (1.1) has a pair of pure imaginary roots and that the elementary divisors are simple (Kamenkov's case). We assume that the linear normalization has already been done and H_2 has the form (1.3); the forms H_m have the form (1.2). We now make a nonlinear canonic transformation

$$(x_1,\ X_1,\ x_2,\ X_2) \to (y_1,\ Y_1,\ y_2,\ Y_2) \tag{3.1}$$

so as to maximally simplify the forms H_3, H_4, etc. The normalizing transformation (3.1) will be sought not by the classical Birkhoff method but by the new Deprit-Hori normalization methods [12, 13]. Then, in the notation of the paper (*) the normalization reduces to the solving of the operator equation

$$DS_m = G_m - F_m \tag{3.2}$$

in each order m relative to the variables. Here S_m are the expansion terms of the generating function of transformation (3.1), F_m are the expansion terms of the new normalized Hamiltonian function F, while the forms G_m are computed by simple formulas in terms of the previously-found forms $H_2, \ldots, H_m, S_3, \ldots, S_{m-1}, F_3, \ldots, F_{m-1}$ (for example, $G_3 = H_3, G_4 = H_4 + {}^1/_2\ \{H_3 + F_3;\ S_3\}$, where the braces denote the operation of computing the Poisson brackets). In (3.2)

$$DS_m = \{S_m; H_2\} = \sum_{j=1}^{2}\left(\frac{\partial S_m}{\partial y_j}\frac{\partial H_2}{\partial Y_j} - \frac{\partial S_m}{\partial Y_j}\frac{\partial H_2}{\partial y_j}\right) \tag{3.3}$$

For convenience in solving the operator Eq. (3.2) we make a linear complex canonic change of variables

$$x_1 = x_1^*, \quad X_1 = X_1^* \tag{3.4}$$

$$x_2 = \frac{1}{\sqrt{2}}(x_2^* + i\delta_2 X_2^*), \quad X_2 = \frac{1}{\sqrt{2}}(i\delta_2 x_2^* + X_2^*) \tag{3.5}$$

Corresponding formulas can be written out for the variables y_j, Y_j, y_j^* and Y_j^*. Then, allowing for the form of function H_2, written down in terms of complex variables, we have

$$D = i\omega_2\left(y_2^*\frac{\partial}{\partial y_2^*} - Y_2^*\frac{\partial}{\partial Y_2^*}\right)$$

This leads to the solving of algebraic equations in the coefficients of the generating function and of the new Hamiltonian function (these functions can also be written in complex variables):

$$\omega_2(\nu_2 - \mu_2)\, s^*_{\nu_1\mu_1\nu_2\mu_2} = -i\,(g^*_{\nu_1\mu_1\nu_2\mu_2} - f^*_{\nu_1\mu_1\nu_2\mu_2})$$

Hence we see that if $\nu_2 \neq \mu_2$, the corresponding terms in the new Hamiltonian function can be annulled. Returning to real variables, we obtain the normal form of the Hamilton-

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*) Markeev, A. P. and Sokol'skii, A. G., Certain computational algorithms for the normalization of Hamiltonian systems. Preprint from the Institute of Applied Mathematics, Akad. Nauk SSSR, № 31, 1976.

ian function

$$F = \frac{1}{2}\delta_2\omega_2(y_2^2 + Y_2^2) + \sum_{m=3}^{M}\sum_{k=0}^{[m/2]} h^{(k)}_{m-2k}(y_2^2 + Y_2^2) + F_{M+1} + \ldots \quad (3.7)$$

Here $h^{(k)}_{m-2k}$ are homogeneous polynomials of degree $m - 2k$ in y_1 and Y_1, while the brackets denote the operation of taking the integer part of a number. It is assumed that the normalization has been carried out up to such an order M relative to the variables that at least one coefficient of the polynomial $h^{(0)}_M$ is nonzero.

Theorem 3.1. Suppose that a canonic system with two degrees of freedom has one zero frequency and simple elementary divisors and that its Hamiltonian function has been reduced to form (3.7). Then, if $h^{(0)}_M$ is a sign-definite function of the variables y_1 and Y_1, the equilibrium position is Liapunov-stable. If $h^{(0)}_M$ is a sign-variable function, the equilibrium position is unstable.

Note 3.1. If $h^{(0)}_M$ is a sign-constant function, the stability question is resolved by higher-order terms.

Note 3.2. The last statement in Theorem 3.1 is proved here under the additional assumption that among the roots of the equation $h^{(0)}_M = 0$ there is a simple one. The theorem remains valid without this assumption.

Corollary 3.1. The equilibrium position is unstable if M is an odd number.

To prove the theorem we pass to the polar coordinates r_j and φ_j by the formulas

$$y_j = \sqrt{2r_j}\sin\varphi_j, \quad Y_j = \sqrt{2r_j}\cos\varphi_j \quad (3.8)$$

The Hamiltonian function (3.7) then becomes

$$F = \delta_2\omega_2 r_2 - \delta_2\omega_2 r_1^{M/2}\Phi(\varphi_1) + \sum_{m=3}^{M}\sum_{k=1}^{[m/2]} \Phi_{m,k}(\varphi_1) r_1^{m/2-k} r_2^{k} + F^{(M+1)} \quad (3.9)$$

where the function $-\delta_2\omega_2\Phi(\varphi_1)$ is obtained from $h^{(0)}_M$ if instead of y_1 and Y_1 we substitute the quantities $\sqrt{2}\sin\varphi_1$ and $\sqrt{2}\cos\varphi_1$, respectively. The functions $\Phi_{m,k}$ are obtained similarly. By $F^{(M+1)}$ we have denoted terms whose orders relative to $\sqrt{r_j}$ are higher than M and which are 2π-periodic in φ_1 and φ_2.

Using the integral $F = H^\circ = \mathrm{const}$ we reduce the system's order by two units and we reduce the investigation of the autonomous system with two degrees of freedom to the investigation of a system with one degree of freedom, but a nonautonomous one. The new Hamiltonian $K = r_2$ is a 2π-periodic function of the new independent variable φ_2. The motion is being analyzed in a fairly small neighborhood of the origin; therefore, we can take $r_1 \sim \varepsilon$ and $r_2 \sim \varepsilon$, where $0 < \varepsilon \ll 1$. In addition, let the initial conditions be such that $H^\circ \sim \varepsilon^{(M+1)/2}$. Then, by solving Eq. (3.9) relative to r_2 and denoting the Hamiltonian of the resulting system by K, we arrive at (2.1) where $\alpha = M/2$, $\alpha_1 = 1/2$, $\tau = 2\pi$, $\varphi = \varphi_1$ and $r = r_1$. Finally, noting that the conditions of sign-definiteness or sign-variability of function $h^{(0)}_M$ are equivalent to the corresponding conditions of the absence or presence of the roots of the equation $\Phi(\varphi) = 0$, which appear in the fundamental Theorem 2.1, we obtain right away both the assertions of Theorem 3.1.

4. Let us consider the case of multiple elementary divisors (Liapunov's case). The quadratic part of the Hamiltonian function (1.1) has the form (1.4). By the same method as in the case of simple elementary divisors we carry out a nonlinear normalization. Instead of change (3.4) we make the change $x_1 = \sqrt{-i\delta_1}x_1^*$ and $X_1 = \sqrt{i\delta_1}X_1^*$. Then the quadratic part of the Hamiltonian function becomes

$$H_2^* = -{}^1/_2 i x_1^{*2} + i\omega_2 x_2^* X_2^*$$

Substituting this expression into the operator Eq. (3.2), we arrive at the problem of solving the system of algebraic equations

$$(\mu_1 + 1) s^*_{\nu_1-1,\,\mu_1+1,\,\nu_2,\,\mu_2} + \omega_2(\nu_2 - \mu_2) s^*_{\nu_1\mu_1\nu_2\mu_2} = -i(g^*_{\nu_1\mu_1\nu_2\mu_2} - f^*_{\nu_1\mu_1\nu_2\mu_2})$$

Examining these equations, it is easy to see that in the new Hamiltonian function we cannot annul only the terms for which $\nu_1 = 0$ and $\nu_2 = \mu_2$. This leads to a normal form of the Hamiltonian, written in real variables

$$F = \frac{1}{2}\delta_1 y_1^2 + \frac{1}{2}\delta_2\omega_2(y_2^2 + Y_2^2) + \sum_{m=3}^{M}\sum_{k=0}^{[m/2]} a_{m-2k,\,k} Y_1^{m-2k} \times \tag{4.1}$$

$$(y_2^2 + Y_2^2)^k + F_{M+1} + \ldots$$

where $a_{m-2k,k}$ are real coefficients. The normalization must be carried out up to an order M such that $a_{M,0} \neq 0$. The coefficients of the normal form are expressed rather simply in terms of the coefficients of the original Hamiltonian function (1.1). For example

$$a_{3,0} = h_{0300} \tag{4.2}$$

$$a_{4,0} = h_{0400} - \frac{1}{2}\delta_1 h_{1200}^2 + \frac{3}{2}\delta_1 h_{2100} h_{0300} - \frac{1}{\omega_2}\delta_2 h_{0210} h_{0201}$$

Theorem 4.1. Suppose that a canonic system with two degrees of freedom has one zero frequency and multiple elementary divisors and that its Hamiltonian function has been reduced to form (4.1). Then

1) if M is odd, the equilibrium position is unstable;
2) if M is even and $\delta_1 a_{M,0} < 0$, the equilibrium position is unstable;
3) if M is even and $\delta_1 a_{M,0} > 0$, the equilibriun position is Liapunov-stable.

To prove the theorem we introduce the polar coordinates r_2 and φ_2 by formulas (3.8) and, as in Sect. 3, we lower the system's order by using the integral $F = H^\circ = \text{const}$, having taken φ_2 as the new independent variable. For the new Hamiltonian function K we obtain the expression

$$K = \frac{1}{\delta_2\omega_2}\left[\frac{1}{2}\delta_1 y_1^2 + a_{M,0} Y_1^M + K^{(M)} + K^{(M+1)}\right] \tag{4.3}$$

$$K^{(M)} = \sum_{m=3}^{M}\sum_{k=1}^{[m/2]} a^*_{m-2k,\,k} Y_1^{m-2k} y_1^{2k} \tag{4.4}$$

$$K^{(M+1)}(y_1, Y_1, \varphi_2, H^\circ) = O\left((y_1^2 + Y_1^2)^{(M+1)/2}\right)$$

In (4.4) the real quantities $a^*_{m-2k,k}$ are obtained by simple formulas from the quantities $a_{i,j}$ $(i,\ j = 1, \ldots, m)$.

Let us prove the first assertion in Theorem 4.1 using the Liapunov's instability theorem [1]. As the Liapunov function we take the sign-variable function

$$V = \delta_2\omega_2(y_1 - M\delta_1 a_{M,0} y_1 Y_1)$$

The derivative of function V taken relative to the equations of motion with the Hamiltonian function (4.3) is

$$dV/d\varphi_2 = Ma_{M,0}(y_1^2 + Y_1^{M-1}) + V^* \tag{4.5}$$

where V^* denotes either terms of higher order in y_1 and Y_1 or terms of the form $y_1^m Y_1^k$, where $3 \leqslant m + k \leqslant M - 1$, but $m \neq 0$. In a fairly small neighborhood of the origin the function (4.5) is sign-definite. This is obvious if we change $y_1 \to \delta^{M-1}y$ and $Y_1 \to \delta^2 Y$, where δ is a fairly small number. Then all the terms from V^* are terms of higher order in δ and the sign of function (4.5) coincides with the sign of $a_{M,0}$. Thus, the first assertion of Theorem 4.1 is proved.

To prove the theorem's last two assertions we make the canonic change of variables

$$y_1 = \sqrt{2} r^{M/(M+2)} \operatorname{Sn} \varphi, \quad Y_1 = \frac{M+2}{2\sqrt{2}} r^{2/(M+2)} \operatorname{Cs} \varphi$$

where φ and r are the new coordinate and momentum, while $\operatorname{Sn} \varphi$ and $\operatorname{Cs} \varphi$ are Liapunov functions [9] defined by the formulas

$$\frac{d \operatorname{Cs} \varphi}{d\varphi} = -\operatorname{Sn} \varphi, \quad \frac{d \operatorname{Sn} \varphi}{d\varphi} = \operatorname{Cs}^{M-1}\varphi, \quad \operatorname{Cs} 0 = 1, \quad \operatorname{Sn} 0 = 0 \tag{4.6}$$

$$\operatorname{Cs}^M \varphi + {}^1/_2 M \operatorname{Sn}^2 \varphi = 1$$

Note that when $M = 4$ we have $\operatorname{Cs} \varphi = \operatorname{cn} \varphi$ and $\operatorname{Sn} \varphi = \operatorname{sn} \varphi \operatorname{dn} \varphi$, where $\operatorname{cn} \varphi$, $\operatorname{sn} \varphi$ and $\operatorname{dn} \varphi$ are elliptic functions with modulus $1 / \sqrt{2}$. The functions $\operatorname{Sn} \varphi$ and $\operatorname{Cs} \varphi$ are periodic with a common period expressed in terms of the gamma-function as follows:

$$T_M = 2\sqrt{\frac{2\pi}{M}}\, \Gamma\left(\frac{1}{M}\right) \Big/ \left(\frac{M+2}{2M}\right)$$

The Hamiltonian function (4.3), written in the variables r and φ, has the form (2.1), where

$$\alpha = 2M \,/\, (M+2), \quad \alpha_1 = 1 \,/\, (M+2), \quad \tau = T_M \tag{4.7}$$

$$\Phi(\varphi) = \frac{\delta_1}{\delta_2 \omega_2} [\operatorname{Sn}^2 \varphi + A \operatorname{Cs}^M \varphi], \quad A = \delta_1 a_{M,0} \left(\frac{M+2}{2\sqrt{2}}\right)^M \tag{4.8}$$

Comparing expression (4.6) with the expression within brackets in (4.8) we see that the function $\Phi(\varphi)$ has no zeros when $\delta_1 a_{M,0} > 0$. For $\delta_1 a_{M,0} < 0$ we obtain $\Phi(\varphi^*) = 0$, where

$$\operatorname{Sn} \varphi^* = -\delta_1 \delta_2 \left[-\left(1 - \frac{AM}{2}\right) \Big/ A\right]^{-1/2}, \quad \operatorname{Cs} \varphi^* = \left(1 - \frac{AM}{2}\right)^{-1/M}$$

$$\Phi'(\varphi^*) = -\frac{2}{\omega_2} \sqrt{-A} \left(1 - \frac{AM}{2}\right)^{(2-M)/2M} \neq 0$$

Now making use of the fundamental theorem in Sect. 2, we at once obtain the last two assertions of Theorem 4.1.

5. To illustrate the applicability of the results described above in mechanical problems, we consider the stability question for the Lagrange solutions of the planar circular restricted three-body problem with a zero mass ratio [14]. The first few terms in the expansion of the Hamiltonian function in a neighborhood of a Lagrange solution are

$$H_2 = \frac{1}{2} q_1^2 - k q_1 q_2 - q_1 p_2 - \frac{5}{8} q_2^2 + q_2 p_1 + \frac{1}{2} p_1^2 + \frac{1}{2} p_2^2 \tag{5.1}$$

$$H_3 = -\frac{7\sqrt{3}}{36} k q_1^3 + \frac{3\sqrt{3}}{16} q_1^2 q_2 + \frac{11\sqrt{3}}{12} k q_1 q_2^2 + \frac{3\sqrt{3}}{16} q_2^3 \tag{5.2}$$

$$H_4 = \frac{37}{128} q_1^4 + \frac{25}{24} k q_1^3 q_2 - \frac{123}{64} q_1^2 q_2^2 - \frac{15}{8} k q_1 q_2^3 - \frac{3}{128} q_2^4 \tag{5.3}$$

$$k = {}^3/_4 \sqrt{3}\,(1 - 2\mu)$$

where μ is the ratio of the mass of the smallest of the main bodies involved to the sum of their masses. When $\mu > \mu^* \approx 0.0385...$ the Lagrange solutions [14] are unstable since the defining equation has roots with positive real part. The stability question for the Lagrange solutions for $0 < \mu \not< \mu^*$, which requires a nonlinear analysis, has been considered in [15 — 17]. For $\mu = 0$ the answer to the stability question follows from simple physical considerations. As a matter of fact, when $\mu = 0$ the problem reduces to an investigation of the stability of the motion of a material point around a fixed attracting center, such a motion is Liapunov-unstable; its period depends upon the initial conditions. However, it is of interest to obtain this same result from a purely formal consideration of a Hamiltonian system with two degrees of freedom along the lines of the results in [15 — 17].

When $\mu = 0$ the frequencies of the motion in a neighborhood of a Lagrange solution have the form: $\omega_1 = 0$ and $\omega_2 = 1$, while the elementary divisors of the defending matrix of the linear system are multiple. The canonic transformation

$$Q = N\xi \tag{5.4}$$

$$Q = \left\| \begin{matrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{matrix} \right\|, \quad N = \left\| \begin{matrix} 0 & \frac{\sqrt{13}}{2} & -\frac{3}{2} & 0 \\ -\frac{4}{3} & -\frac{3\sqrt{3}}{2\sqrt{3}} & \frac{\sqrt{3}}{2} & \frac{4}{\sqrt{13}} \\ -\frac{1}{6} & \frac{3\sqrt{3}}{2\sqrt{13}} & -\frac{\sqrt{3}}{2} & \frac{5}{2\sqrt{13}} \\ \frac{\sqrt{3}}{2} & \frac{5}{2\sqrt{13}} & -\frac{3}{2} & -\frac{3\sqrt{3}}{2\sqrt{13}} \end{matrix} \right\| \quad \xi = \left\| \begin{matrix} x_1 \\ x_2 \\ X_1 \\ X_2 \end{matrix} \right\|$$

reduces the quadratic part (5.1) of the Hamiltonian function to the form (1.4), while $\delta_1 = -1$ and $\delta_2 = 1$. The forms H_3 and H_4 are written in form (1.2) wherein the coefficients $h_{\nu_1\mu_1\nu_2\mu_2}$ are simply expressed in terms of the coefficients of forms (5.2) and (5.3) and of the coefficients of transformation (5.4). By the nonlinear normalization method described in Sect. 4 the Hamiltonian function is brought to form (4.1), where (see (4.2) and (4.3))

$$a_{3,0} = 0, \quad a_{4,0} = \frac{81}{104}(13 - 4\sqrt{3})$$

Since $\delta_1 a_{4,0} < 0$, according to the second assertion of Theorem 4.1 we get that the Lagrange solutions are unstable when $\mu = 0$.

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